[TOPIC: DIFFERENTIAL CALCULUS – INTRODUCTION]

Calculus is an incredibly powerful tool that sees extensive application in almost all areas of physical life, from determining the fight paths of projectiles and planets to modelling the spread of disease, the growth of human population or the movement of oceans. At its heart, we are attempting to study rates of change.

DEFINITION

Before we deal with curves, recall that the gradient of a straight line describes the amount by which y will change when you change x by 1. Put another way, it is the rate of change of y with respect to x. The idea of (differential) calculus is to generalise this idea of gradients to more general curves – the vast majority of useful curves (and curves in general) are not straight lines! – and thus be able to quantify the rate of change of y with respect to x for more general functions. In order to do this, we use tangent lines. In particular, we pick a point on a curve and use the gradient of the line tangent to the curve at that point to describe the rate of change of y with respect to x at that point. Differentiation is essentially the process undertaken to determine the gradient of the tangent to a curve at a given point.

Now, given a general function and a point A on the curve, we have to find the gradient of the tangent. We almost immediately run into a problem – ordinarily, to calculate a gradient, we require two points. However, we only have one, being the point where the tangent contacts the curve. To solve this, we pick another point B on the curve a small distance away – we usually call this distance “h” – and draw a secant between A and B. It’s not a tangent, and sometimes it’s not even close. But, if we move B closer to A – which is equivalent to making h smaller – we find that the secant line approaches the tangent line. When A and B are very close together – i.e. h is very small – the secant and tangent are almost the same line. So, in the limit as h approaches zero, the gradient of the secant should approach the gradient of the tangent.

So, we have a method: pick a point A, pick a second point B that isn’t too far away, draw a secant and then move B closer and closer to A; as they get very close, the gradient of the secant approaches the gradient of the tangent. This process is referred to in the HSC course as “differentiation from first principles”.

NOTATION

There are a few different ways of denoting derivatives, and it is extremely important that you understand each of them and where to use them appropriately. When given a function f(x), we can denote the derivative as f’(x) (read: “f-dash of x”) or, less commonly, df(x)/dx – I prefer the first option because it’s more space efficient, and you will most often see this form in textbooks and exams. If given a curve like y = ……… then we can denote the derivative as y’ (read: “y-dash”) or dy/dx (read: “d-y, d-x”).

FINDING DERIVATIVES

You can differentiate some simple functions using the definition (first principles), but after the first couple is quickly becomes clear that it is tedious and often difficult if not impossible. There must be a better way! There is, using theorems about differentiation. In particular, arithmetic properties of differentiation (sums and multiples of derivatives) and the differentiation rules: power, chain, product, quotient. It is imperative that you remember these rules! Once you learn them here, it will be assumed knowledge for the rest of your HSC (and, trust me, you will need them all).

APPLICATIONS

There are a few main applications of differential calculus which are studied in the year 11 course. They include finding equations of tangents and normal to curves, identifying and classifying stationary points and finding points of inflexion.

TANGENTS AND NORMALS

Remember (and I cannot stress this enough) that the derivative describes the gradient of the tangent to a curve. So, to find the equation of the tangent, we simply need to find the derivative and use the point-gradient formula. For normals, we need only remember the fact that the normal is perpendicular to the tangent, so the product of the gradients is –1.

STATIONARY POINTS

When finding and classifying stationary points there are a number of tips which can make your (and your marker’s!) life easier.

[A] READ THE QUESTION: Note whether the question asks for “coordinates”, as this means both x and y values, and whether the question wants you to find points of inflexion – some questions will explicitly ask you NOT to do so.

[B] THINK ABOUT WHICH TEST TO USE: Use either the first or second derivative tests depending on which is less work. The second derivative test is often easier, particularly for functions such as polynomials, however, sometimes functions are simply too time-consuming to differentiate a second time, and the first derivative test may be more appropriate. It is purely up to you which you prefer under what circumstances, and you will quickly build up experience in deciding this. Personally, I prefer the second derivative test in almost all cases, but it depends on what you are confident with.

If you use the first derivative test,

[C] WRITE THE VALUES OUT: When testing the derivatives on either side of a stationary point (the first derivative test), it is recommended to write the value of the derivative, not just the sign, if for no other reason than to convince your marker that you actually checked the value and didn't just guess. Remember, you are allowed to use a calculator and the memory functions on your calculator can be extremely useful and save a lot of time when substituting several different values into the same equations.

[D] DRAW THE LINES: It is common practice to draw lines below each value of the derivative, with the gradient of the line corresponds to the sign of the derivative. I fully recommend doing this! The shape traced out by these lines corresponds to the nature of the stationary point. This is a visual way to ensure your conclusions are correct.

[E] CHOOSE TEST POINTS CAREFULLY: When testing the first derivative, it is imperative that one chooses test points that:

* Have a well-defined value of the derivative;
* Are not other stationary points; and
* Do not have other stationary points between the chosen point and the stationary point being tested.

Numbers like –1, 0 and other integers are the easiest to use as they are usually easy to substitute into equations, but you can use whatever points you want, as long as they follow the above rules.

Finally, whichever test you’ve used,

[F] BE EXPLICIT: Make your conclusions clear. Questions like these require a lot of calculation, and they can get quite messy. Once your calculations are complete, write clearly at the bottom of the question where the stationary points are (both coordinates if required) and whether they are local maxima, local minima or horizontal points of inflexion. Underline it, put it in a box, just make it obvious. This makes it easier for a marker to read and for you to follow.

Points of Inflexion

Recall that a point of inflexion of a function f’’(x) is a point where f’’(x) = 0 AND there is a change in concavity. The second part of this definition is absolutely CRUCIAL, and so, so many students forget this point entirely because the functions dealt with in this course are chosen specifically because they are well-behaved. You will lose marks if you do not check for a change in concavity! Please do it. Also note that the same rules for picking test points when checking stationary points applies to testing for changes in concavity around possible points of inflexion.

Conclusion

What do you need to be able to do to successfully complete your examinations? You must be able to

* Differentiate simple functions from first principles;
* Use important differentiation rules (arithmetic properties, power rule, chain rule, product rule, quotient rule) and be able to differentiate functions successfully and efficiently;
* Use the derivative to find gradients and equations of tangents and normals to curves at given points;
* Use the derivatives to locate and determine the nature of stationary points and points of inflexion – learn both methods for determining the nature of stationary points!
* Use the derivatives to describe the behaviour of curves, such as “increasing” or “decreasing” (at a point or otherwise) and concavity;
* Apply the derivative to physical problems – understanding the derivative as a rate of change of a quantity; rate of change/optimisation (max/min) problems;
* Interpret the derivative geometrically – understanding the derivative as the gradient of the tangent, drawing graphs of the first and second derivative given the graph of a general function.

You now have a formula sheet for this course which contains many of the basic formulae, including for this topic. This is both a blessing and a curse – while you may not have to remember the formulae precisely, questions will focus less on remembering the requisite formulae and more on how to apply them. As such it is imperative that you understand not only that a formula exists but also when and where it should be used!